Uniform Stability for Impulsive Delay Differential Equations

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Abstract

In this paper we obtain sufficient conditions for the stability of the zero solution of the delay differential equations with impulses

$$x'(t) + F\big(t, x(.)\big) = 0, \qquad t \geq 0, \qquad t \neq \tau_k$$

$$x({\tau_k}^{+}) - x(\tau_k) = I_k\big(x(\tau_k)\big), \qquad k = 1,2,\ldots, \lim_{k\to\infty} \tau_k = 0$$

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Introduction

The theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulses, but also represents a more natural framework for mathematical modeling of many real world phenomena. Most dynamical systems- physical,social, biological, engineering are often conveniently expressed in the form of such differential equations. The number of publications dedicated to its investigation has grown constantly in the recent years and a well developed theory has taken in shape. See monographs [1,10] and references cited therein. However, concerning the stability of delay differential equations with impulses, the results are relatively scarce. In section 3, we shall give sufficient conditions for the zero solution of (1) to be uniformly stable.

Aim of the Study

The aim of this paper is to study the stability of a scalar impulsive delay differential equation. Some new results on stability are established. The results in this paper improve and extend several known results in literature.

Review of Literature

The study of certain ordinary differential equations with impulses was initiated in the 1960's by Milman and Myshkis. They investigate the stability of the zero solution of differential equations with fixed moments of impulse actions by using the second Lyapunov method.In 1989, V. Lakshmikantham, investigate the stability theory relative to a given solution of impulsive differential systems that played an important role in the development of complicated mechanism. The 3/2 stability theorem for one – dimensional delay differential equation was established by T. Yoneyama in 1990 [12]. In 2000, G. Ballinger and X.Liu study the existence, uniqueness and boundedness results for impulsive delay differential equations. In 2007 Yu Zhang and Jitao Sun [15], investigates some sufficient criteria of stability , asymptotic stability and practical stability for impulsive functional differential equations in which the state variables on the impulses are related to the time delay are provided by using Lyapunov functions and Razumikhin techniques. The stability of ordinary differential equations with impulses has been extensively studied in the literature. See monographs [1,10] and references cited therein, the theory and application of impulsive differential equations and delay differential equations have been extensively developed. In recent years, theory of impulsive delay differential equations has been an object of active research see [3,4,10,13,14]

Definitions and Assumptions

Let $g: [0, \infty) \to R$ be a non decreasing continuous function such that $g(t) \leq t$ for all $t \geq 0$. For $H > 0$ and $t \geq 0$, let $C_H(t)$ be the set of continuous functions ϕ, $\phi: [g(t), t] \to R$ such that

$$\|\phi\|_t = \sup_{s\in[g(t),t]} |\phi(s)| < H$$

Consider the delay differential equation with impulses

$$\left.\begin{array}{l} x'(t) + F(t, x(.)) = 0,\ t \geq 0,\ t \neq \tau_k \\ x(\tau_k^{+}) - x(\tau_k) = I_k(x(\tau_k)),\ k \in N = \{1,2,.,\} \end{array}\right\}$$
(1)

where $F(t,\phi)$ is a continuous functional of $t \geq 0$, $\phi \in C_H(t)$ and $x'(t)$ denote left handderivative of $x(t)$. $I_k: R \to R$, $0 \leq \tau_1 < \tau_2 < \tau_3 < \ldots \tau_k < \ldots$. are fixed points with $\lim_{k\to\infty} \tau_k = \infty$. We assume that $F(t,0) \equiv 0$ and $I_k = 0$, so that $x(t) \equiv 0$ is a solution of (1), which we call the zero solution.

Definition 1

For $t_0 \geq 0$, $\phi \in C(t_0)$, a function $x(t)$ defined on $[g(t_0), \infty)$ is said to be a solution of (1) and is denoted by $x(t, t_0, \phi)$, if it satisfies the initial value condition

$x(t) = \phi(t)$ for $g(t_0) \leq t \leq t_0$ (2)

and the following assumptions are satisfied

(i) $x(t)$ satisfies (2) on $[g(t_0), t_0]$;

(ii) $x(t)$ is absolutely continuous on each interval (t_0, τ_{k_0}), (τ_k, τ_{k+1}), $k \geq k_0$ where $k_0 \in N$ is such that $\tau_{k_0-1} \leq t_0 \leq \tau_{k_0}$;

(iii) $x(t)$ satisfies (1)

Definition 2

The zero solution of (1) is said to be stable if for each $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that $\phi \in C_\delta(t_0)$ implies $|x(t, t_0, \phi)| < \epsilon$ for all $t \geq t_0$.

Definition 3

The zero solution of (1) is uniformly stable if δ is independent of t_0.

We assume that there exist a continuous function $a : [0, \infty) \to [0, \infty)$ such that

$-a(t)M_t(-\phi) \leq F(t,\phi) \leq a(t)M_t(\phi)$

for $t \geq 0$ and $\phi \in C_H(t)$ (3)

where $M_t(\phi) = max\left\{0, \sup_{s\in[g(t),t]} \phi(s)\right\}$

When the impulses of (1) are removed, i.e. $I_k(x) \equiv 0$ for any $x \in R$ and $k \in N$, (1) reduced to a scalar delay differential equation without impulsive effects.

$$x'(t) + F(t, x(.)) = 0,\ t \geq 0 \quad (4)$$

Lemma 1

Assume that $g(t) < t$ and (3) hold. $t_0 \geq 0$ and $\phi \in C_H(t_0)$. Then the solution $x(t, t_0, \phi)$ of (4) satisfying initial condition (2) on $[t_0, \infty)$ and satisfies $\|x\|_t \leq \|x\|_{t_0} exp\left(\int_{t_0}^{t} a(s)ds\right)$ for $t_0 \leq t < \infty$

Proof

A solution $x(t, t_0, \phi)$ of (4) with (2) exists on $[g(t_0), t_0 + \alpha)$, where $\alpha > 0$. By continuation theorem of solutions, $x(t)$ can be continued on $[g(t_0), T)$ where $T \geq t_0 + \alpha$. Now, we prove $T = \infty$. Otherwise, suppose that $T < \infty$. Then $x(t)$ is a non-continuable solution on $[t_0, \infty)$.

By (3) and (4) we obtain that for $t_0 \leq t < T$

$(x(t))' = -F(t, x(.)) \leq$

$a(t)\, max\left\{0, \sup_{s\in[g(t),t]} -x(s)\right\}$ (5)

On the other hand, similarly we can obtain that

$-(x(t))' \leq a(t)\, max\left\{0, \sup_{s\in[g(t),t]} x(s)\right\}$ (6)

Combining (5) and (6), it is easy to get

$|x'(t)| \leq a(t)\|x\|_t$ for $t_0 \leq t < T$ (7)

Integrating (7) from t_0 to t and simplifying it, we obtain

$\|x\|_t \leq \|x\|_{t_0} + \int_{t_0}^{t} a(s)\|x\|_s ds$ for $t_0 \leq t < T$

By Gronwall's inequality, we have

$\|x\|_t \leq \|x\|_{t_0} exp\left(\int_{t_0}^{t} a(s)ds\right)$ for $t_0 \leq t < T$

which implies that $x(t)$ is bounded on $[t_0, T)$. Since $x(t)$ is absolutely continuous on $[t_0, T)$, it is uniformly continuous on $[t_0, T)$, which implies $lim_{t\to T^-} x(t)$ exists. Set $x(T) = lim_{t\to T^-} x(t)$. Consider initial value problem (4) with condition

$\phi(t) = x(t)$, $g(T) \leq t \leq T$ (8)

Then, as the preceding arguments, (4) with (8) has a solution $x(t)$ on $[T, T + \alpha_1)$, $\alpha_1 > 0$, which contradicts the fact that $x(t)$ is a non-continuable solution on $[t_0, T)$. Thus $T = \infty$.

The proof of Lemma 1 is complete.

Theorem 1

Assume that (3) hold for any $t_0 \geq 0$ and $\phi \in C(t_0)$. Then the initial value problem with (1) with (2) has a solution on $[t_0, \infty)$

Proof

Without loss of generality, we suppose that $t_0 \leq \tau_k$, $k \in N$. By Lemma 1, (1) with (2) has a solution $x(t, t_0, \phi)$ satisfying $x(t) = \phi(t), g(t_0) \leq t \leq t_0$ on $[t_0, \infty)$. Consider the initial value problem (1) with condition

$$\phi_1(t) = \begin{cases} x(t), & g(\tau_1) \leq t \leq \tau_1 \\ x(\tau_1) + I_1(x(\tau_1)), & t = \tau_1 \end{cases} \quad (9)$$

By Lemma 1 again (4) with (8) has a solution on $[\tau_1, \tau_2)$. Using similar arguments repeatedly, we can obtain a sequence of solutions $x_k(t, \tau_k, \phi_k)$ on $[\tau_k, \tau_{k+1})$, $k = 0,1,2, \ldots$, where $\tau_0 = t_0, \phi_0 = \phi$ which satisfies

(i) $x(t)$ is defined on $[g(t_0), t_0]$ and $x(t) = \phi(t), g(t_0) \leq t \leq t_0$ and satisfies (4) on $[t_0, \tau_1)$;

(ii) $x_k(t)$ is defined on $[g(\tau_k), \tau_k)$ and $x_k(t) = \phi_k(t)$, $g(\tau_k) \leq t \leq \tau_k$.

Let $x(t) = \begin{cases} \phi(t), & g(t_0) \leq t \leq t_0 \\ x_k(t), & g(\tau_k) \leq t \leq \tau_k,\ k \in N \end{cases}$

Obviously, $x(t)$ satisfies (3) on $(\tau_k, \tau_{k+1}), k \in N$ and for any τ_k, $x(\tau_k^{+}) - x(\tau_k) = I_k(x(\tau_k))$. Hence $x(t)$ is a solution of (1) with (2) on $[t_0, \infty)$. The proof of Theorem 1 is complete.

Let $g: [0, \infty) \to R$ be a non decreasing continuous function satisfying $g(t) \to \infty$ as $t \to \infty$ with $g(t) \leq t$ for all $t \geq 0$. Thus $g(t)$ is bounded. Hence there exists T such that for all $t \geq T, g(t) \geq 0$. Note that $g(g(t)) \leq g(t) \leq t$ for all $t \geq T$

For $t \geq 0$, let $g^{-1}(t) = sup\{s \geq T: g(s) = t\}$. Then $g^{-1}(t)$ is a piecewise continuous function satisfying $t \leq g^{-1}(t)$ for all $t \geq T$. Define $g^{-1}(g^{-1}(t)) = g^{-2}(t)$

Uniform Stability

In this section, we study stability of the zero solution of (1). For this, first we establish a fundamental lemma which helps to reduce stability of impulsive delay differential equation to the problem of non impulsive delay differential equation.

We define the function

$J_k(U) = \frac{U}{U+I_k(U)}$ where $U \in R, U \neq 0, k \in N$

Lemma 2

If $x(t) = x(t, t_0, \phi)$ *is a solution of (1) on* $[t_0, \infty)$, *then*

$y(t) = y(t, t_0, \phi)$ (10)

is an absolutely continuous solution of the following equation:

$y'(t) + \prod_{t_0 \leq \tau_k < t} J_k(x(\tau_k))F(t, x(.)) = 0,\ t \geq t0 a.e$ (11)

Proof

As $x(t)$ is absolutely continuous on each interval of $(t_0, \tau_{k_0}),\ (\tau_k, \tau_{k+1}),\ k \geq k_0,\ x(t)$ is absolutely continuous there.

Besides, for any $\tau_k \geq t_0$

$$y(t_k{}^+) = \prod_{t_0 \leq \tau_i < \tau_k} J_i(x(\tau_i))x(\tau_k{}^+)$$

$$= \prod_{t_0 \leq \tau_i < \tau_k} \frac{x(\tau_i)}{x(\tau_i) + I_i(x(\tau_i))} x(\tau_k{}^+)$$

$$= \prod_{t_0 \leq \tau_i < \tau_k} \frac{x(\tau_i)}{x(\tau_i) + I_i(x(\tau_i))} x(\tau_k) = y(\tau_k)$$

and $y(t_k{}^-) = \prod_{t_0 \leq \tau_j < \tau_{k-1}} J_i(x(\tau_i))x(\tau_k) = y(\tau_k)$

Which implies that $y(t)$ is continuous on $[t_0, \infty)$. It is easy to prove that $y(t)$ is absolutely continuous on $[t_0, \infty)$. We can see that $y(t)$ is solution of (11) on $[t_0, \infty)$. The proof of lemma 2 is complete.

Theorem 2

Assume that (3) hold and there exist a sequence $\{b_k\}$ *of positive constants and constant* $H > 0$ *such that*

$1 \leq \frac{x}{x+I_k(x)} \leq \frac{1}{b_k}$ *for* $0 < |x| < H$ *and* $k \in N$ (12)

and

$\int_{g(t)}^{t} \prod_{g(s) \leq \tau_k \leq s} \frac{1}{b_k} a(s)ds \leq \frac{3}{2}$ *for all* $t \geq 0$ (13)

Then the zero solution of (1) is uniformly stable.

Proof

Let $\lambda = \sup_{t \geq 0} \int_{g(t)}^{t} \prod_{g(s) \leq \tau_k \leq s} \frac{1}{b_k} a(s)ds$ (14)

For any $\epsilon > 0\ (\epsilon < H)$, choose $\delta = \frac{1}{2}\epsilon e^{-2\lambda}$.

We prove that for any $t_0 \geq 0$ and $\phi \in C_\delta(t_0)$, $x(t) = x(t, t_0, \phi)$ satisfies $|x(t)| < \epsilon$ for all $t \geq t_0$

By using Lemma 2, in view of (12), we only need to show that

$|x(t)| = \prod_{t_0 \leq \tau_k < t} J_k(x(\tau_k))^{-1}|y(t)| \leq |y(t)| < \epsilon$ for all $t \geq t_0$ (15)

First, we prove that

$|y(t)| < \epsilon$ for $t_0 \leq t \leq g^{-2}(t_0)$ (16)

In fact for any $t_0 \leq t \leq g^{-2}(t)$ from (11), (12) and (3), we find

$(y(t))' = -\prod_{t_0 \leq \tau_k < t} J_k(x(\tau_k))F(t, x(.))$ (17)

$\leq \prod_{t_0 \leq \tau_k < t} \frac{1}{b_k} a(t) \max\left\{0, \sup_{g(t) \leq s < t} -x(s)\right\}$

$$= \prod_{t_0 \leq \tau_k < t} \frac{1}{b_k} a(t) \max\left\{0, \sup_{g(t) \leq s < t} - \prod_{t_0 \leq \tau_k < s} J_k(x(\tau_k))^{-1} y(s)\right\}$$

(18)

$$\leq \prod_{g(t) \leq \tau_k < t} \frac{1}{b_k} a(t) \max\left\{0, \sup_{g(t) \leq s < t} -y(.)\right\}$$

On the other hand, from (3) and (12), for $t_0 \leq t \leq g^{-2}(t_0)$

$$-(y(t))' = \prod_{t_0 \leq \tau_k < t} J_k(x(\tau_k))F(t, x(.))$$

$$\leq \prod_{t_0 \leq \tau_k < t} J_k(x(\tau_k))a(t) \max\left\{0, \sup_{g(t) \leq s < t} x(s)\right\}$$

$$\leq \prod_{g(t) \leq \tau_k < t} \frac{1}{b_k} a(t) \max\left\{0, \sup_{g(t) \leq s < t} y(s)\right\} \quad (19)$$

From (18) and (19), we find

$$|y(t)'| \leq \prod_{g(t) \leq \tau_k < t} \frac{1}{b_k} a(t) \sup_{g(t) \leq s \leq t} |y(s)|$$

$= \prod_{g(t) \leq \tau_k < t} \frac{1}{b_k} a(t)\|y\|_t$, for $t_0 \leq t \leq g^{-2}(t_0)$ (20)

Integrating (20) from t_0 to t, then simplifying it, we obtain

$$|y(t)| \leq |y(t_0)| + \int_{t_0}^{t} \prod_{g(s) \leq \tau_k < s} \frac{1}{b_k} a(s)\|y\|_s ds, \text{ for } t_0 \leq t \leq g^{-2}(t_0)$$

It is easy from the last inequality to see that

$$\|y\|_t \leq \|\phi\|_{t_0} + \int_{t_0}^{t} \prod_{g(s) \leq \tau_k < s} \frac{1}{b_k} a(s)\|y\|_s ds, \quad \text{for } t_0 \leq t \leq g^{-2}(t_0)$$

By Gronwall's inequality and using (14) we have that

$$\|y\|_t \leq \|\phi\|_{t_0} exp\left(\int_{t_0}^{t} \prod_{g(s) \leq \tau_k < s} \frac{1}{b_k} a(s)ds\right)$$

$$= \|\phi\|_{t_0} exp\left(\int_{t_0}^{g^{-1}(t_0)} \prod_{g(s) \leq \tau_k < s} \frac{1}{b_k} a(s)ds + \int_{g^{-1}(t_0)}^{t} \prod_{g(s) \leq \tau_k < s} \frac{1}{b_k} a(s)ds\right)$$

$\leq \|\phi\|_{t_0} e^{2\lambda} < \epsilon$

Thus (16) is proved.

Next, we prove that

$|y(t)| < \epsilon$ for all $t \geq g^{-2}(t_0)$ (21)

If (21) were not true, there exists a T $\geq g^{-2}(t_0)$ such that $y(T) = \epsilon$ and $|y(t)| < \epsilon$ for $t_0 \leq t < T$.

Without loss of generality, let $y(T) = \epsilon$. For $y(T) = -\epsilon$ the proof is similar and it is omitted, Hence $y(t) < y(T)$ *for* $g^{-2} < t < T$.

We claim $y(g(T)) \leq 0$. Otherwise there is a $\sigma > 0$ such that $y(t) > 0$ for $g(T) - \sigma < t < g(T)$. Thus $y(g(T)) > 0$ for $g^{-1}(g(T) - \sigma) < t < T$. In view of (10), $x(g(t)) > 0, g^{-1}(g(T) - \sigma) < t < T$, By (3), we have that for $g^{-1}(g(T) - \sigma) < t < T$

$$(y(t))' = -\prod_{t_0\le\tau_k<t} J_k(x(\tau_k))F(t,x(.)) \le 0$$

Hence $y'(t) \le 0$. From the fact that $y(t) > 0$ on $[g(t)-\sigma, T]$, it is easy to see that $y'(t) \le 0\ (a.e)$ in $g^{-1}(g(T)-\sigma) < t < T$ which contradicts the definitiin of T. Therefore, $y(g(T)) \le 0$. So there is a $T_0 \in [g(T), T]$ satisfying $y(T_0) = 0$. On the other hand, we have that

$$(y(t))' = -\prod_{t_0\le\tau_k<t} J_k(x(\tau_k))F(t,x(.))$$
$$\le \prod_{t_0\le\tau_k<t} J_k(x(\tau_k))a(t)\max\left\{0, \sup_{g(t)\le s\le t} -x(s)\right\}$$

$$\le \prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t)\max\left\{0, \sup_{g(t)\le s\le t} -y(s)\right\} \qquad (22)$$
$$< \epsilon\prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t), \quad \text{for } t_0 \le t \le T \qquad (23)$$

As $g(t) \le g(T) \le T_0$ for $T_0 \le t \le T$, integrating (23) from $g(t)$ to T_0, we obtain

$$-y(g(t)) < \epsilon\int_{g(t)}^{T_0}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds, \quad \text{for } T_0 \le t \le T$$

Substituting this into (22), we obtain that for $T_0 \le t \le T$

$$(y(t))' < \epsilon\prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t)\int_{g(t)}^{T_0}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds \qquad (24)$$

Let $\alpha = \int_{T_0}^{T}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds$

Since $T_0 \ge g(T)$, in view of (13), $\alpha < \frac{3}{2}$. We will complete the proof of the theorem if $y(T) < \epsilon$ can be proved because this fact contradicts $y(T) = \epsilon$. First, Suppose $\alpha \le 1$. Integrating (24) from T_0 to T, we have

$$y(t) < \epsilon\int_{T_0}^{T}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)\int_{g(s)}^{T_0}\prod_{g(\xi)\le\tau<\xi}\frac{1}{b_k}a(\xi)d\xi ds$$
$$\le \epsilon\int_{T_0}^{T}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)\left[\int_{g(s)}^{s}\prod_{g(\xi)\le\tau_k<\xi}\frac{1}{b_k}a(\xi)d\xi - \int_{T_0}^{s}\prod_{g(\xi)\le\tau_k<\xi}\frac{1}{b_k}a(\xi)d\xi\right]ds$$
$$\le \epsilon\int_{T_0}^{T}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)\left[\frac{3}{2} - \int_{T_0}^{s}\prod_{g(\xi)\le\tau_k<\xi}\frac{1}{b_k}a(\xi)d\xi\right]ds$$
$$\le \epsilon\left[\frac{3}{2} - \frac{1}{2}\int_{T_0}^{T}\frac{d}{ds}\left(\prod_{g(\xi)\le\tau_k<\xi}\frac{1}{b_k}a(\xi)d\xi\right)^2 ds\right] \le \epsilon$$

This is a contradiction.

Next suppose $\alpha > 1$. Then there exists $T_1 \in (T_0, T]$ such that $\int_{T_1}^{T}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds = 1$

Integrating (23) from T_0 to T_1, we obtain

$$y(T_1) < \epsilon\int_{T_0}^{T_1}\prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t)dt \qquad (25)$$

Integrating (24) from T_1 to T and using (25), we get

$$y(t) < \epsilon\int_{T_0}^{T_1}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds + \epsilon\int_{T_1}^{T}\prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t)\int_{g(t)}^{T_0}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)dsdt$$
$$\le \epsilon\left(\int_{T_1}^{T}\prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t)dt\right)\left(\int_{T_0}^{T_1}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds\right) + \int_{T_1}^{T}\prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t)\int_{g(t)}^{T_0}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)dsdt$$
$$= \epsilon\int_{T_1}^{T}\prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t)\int_{g(t)}^{T_1}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)dsdt$$
$$= \in\int_{T_1}^{T}\prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t)\left(\int_{g(t)}^{t}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds - \int_{T_1}^{t}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds\right)dt$$
$$\le \in\int_{T_1}^{T}\prod_{g(t)\le\tau_k<t}\frac{1}{b_k}a(t)\left(\frac{3}{2} - \int_{T_1}^{t}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds\right)dt$$
$$= \epsilon\left[\frac{3}{2} - \frac{1}{2}\int_{T_1}^{T}\frac{d}{dt}\left(\int_{T_1}^{T}\prod_{g(s)\le\tau_k<s}\frac{1}{b_k}a(s)ds\right)^2 dt\right] = \epsilon$$

This is a contradiction. The proof of Theorem 2 is complete.

Applications

Example 1

Let $f: C_H \to R$ be an increasing continuous function such that $0 < xf(x) \le x^2$ for $x \ne 0$, and $|x| \le H$. Consider the impulsive delay differential equation

$$\left.\begin{aligned} x'(t) &= -a(t)f\left(x(g(t))\right), t \ne \tau_k \quad, t \ge 0 \\ x(\tau_k{}^{+}) - x(\tau_k) &= I_k(x(\tau_k)), k \in N \end{aligned}\right] (26)$$

is uniform stable.

From the results of theorem 2, we have if (13) holds, then zero solution of (26) is uniformly stable.

We give concrete example of above example.

Example C1

Let α be a positive constant such that $e^{-3/2} < \alpha \le 1$. Then the zero solution of $x'(t) = -\frac{1}{t+1}x(\alpha t)$ is uniformly stable.

Conclusion

In this paper, we have considered the uniform stability of impulsive delay differential equation. By using Lyapunov method, we have got result for the uniform stability of zero solution of impulsive delay differential equation.

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